Lecture 3 answers to exercises: Determinants and transformations

1. $(a.)$ $2\cdot 4-3\cdot(-1)=11$

   $(b.)$ $-5\cdot 2-1\cdot 0=-10$

   $(c.)$ Of this matrix we cannot compute the determinant because it is not square.

   $(d.)$ $-5\cdot7\cdot1+1\cdot(-2)\cdot3+(-1)\cdot1\cdot0-(-1)\cdot7\cdot3-1\cdot1\cdot1-(-5)\cdot(-2)\cdot0=-35-6+21-1=-21$

2. $$x=\frac{\begin{vmatrix}2&5&-2\\6&4&-1\\1&2&3\end{vmatrix}}{\begin{vmatrix}4&5&-2\\3&4&-1\\-1&2&3\end{vmatrix}}=\frac{-83}{-4}=20\frac{3}{4}\qquad y=\frac{\begin{vmatrix}4&2&-2\\3&6&-1\\-1&1&3\end{vmatrix}}{\begin{vmatrix}4&5&-2\\3&4&-1\\-1&2&3\end{vmatrix}}=\frac{42}{-4}=-10\frac{1}{2}$$

   $$z=\frac{\begin{vmatrix}4&5&2\\3&4&6\\-1&2&1\end{vmatrix}}{\begin{vmatrix}4&5&-2\\3&4&-1\\-1&2&3\end{vmatrix}}=\frac{-57}{-4}=14\frac{1}{4}$$

3. Yes. Just think of a matrix and apply it to the zero vector. The outcome of all components are zeros.

4. We find the desired matrix by multiplying the two matrices for the two parts. The matrix for reflection in $x+y=0$ is $\begin{bmatrix}0&-1\\-1&0\end{bmatrix}$, and the matrix for rotation of $45°$ about the origin is $\begin{bmatrix}\frac{1}{2}\sqrt{2}&-\frac{1}{2}\sqrt{2}\\\frac{1}{2}\sqrt{2}&\frac{1}{2}\sqrt{2}\end{bmatrix}$. So we compute (note the order!):

   $$\begin{bmatrix}\frac{1}{2}\sqrt{2}&-\frac{1}{2}\sqrt{2}\\\frac{1}{2}\sqrt{2}&\frac{1}{2}\sqrt{2}\end{bmatrix}\begin{bmatrix}0&-1\\-1&0\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\sqrt{2}&-\frac{1}{2}\sqrt{2}\\-\frac{1}{2}\sqrt{2}&-\frac{1}{2}\sqrt{2}\end{bmatrix}$$

5. A must be the zero matrix. This is true because the vectors $\begin{bmatrix}2&3&2\end{bmatrix}$, $\begin{bmatrix}1&0&2\end{bmatrix}$, and $\begin{bmatrix}0&2&4\end{bmatrix}$ are linearly independent. If a linear transformation maps three linearly independenty vectors in $3D$ onto the origin, then it maps all vectors (and points) onto the origin.

   To verify that the three vectors are indeed linearly independent, we compute the determinant $\begin{vmatrix}2&1&2\\3&0&2\\2&2&4\end{vmatrix}$ and observe that is it not zero.

6. $A=\begin{bmatrix}\frac{1}{2}&0\\0&0\end{bmatrix}$

7. We observe what it does with the two base vectors, for instance. It rotates them over $-45°$ and stretches them in length by a factor $\sqrt{2}$. This observation is generally true: any other vector is also rotated and stretched in the same way.

8. The matrix has determinant zero, so it will be a projection. Also, it gives any point the same x and y-coordinates, so it is a projection onto the line $y = x$. It does not project orthogonally, but maps points further away from the origin than their orthogonal projection. (More specifically, points on lines of the form $y = -x + d$ project to the same point on $y = x$, namely the point $(2d, 2d)$.)

9. c influences how much a vector is 'pushed sideways', or, how much a square is sheared. When $c = 0$, there is no sideways pushing, when $c = 1$, the sideways pushing is as much as the y-coordinate, and in general, it is as much as c times the y-coordinate.

10. $\begin{bmatrix} -1 & 0 & 6 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 3 & 0 & -2 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

13. We observe that the two points that are the result of the mapping are the two base vectors. If our information would be that $(0, 1)$ were mapped to $(2, 3)$ and that $(1, 0)$ were mapped to $(9, 7)$, then the matrix would be easy to write down: $\begin{bmatrix} 9 & 2 \\ 7 & 3 \end{bmatrix}$.

    But we are given the 'opposite direction' of the mapping .... But that means that we need the inverted matrix! So the answer is
    $\frac{1}{13} \begin{bmatrix} 3 & -2 \\ -7 & 9 \end{bmatrix}$.

14. We use the cofactors of the bottom row entries to determine the value of the determinant. Since for a matrix in homogeneous coordinates, only the last entry in the last row is non-zero, we only need the cofactor of this one entry. The cofactor is the determinant of the top-left 3×3 determinant, which is exactly the part that describes the linear transformation part of the affine transformation. Hence, the determinant of a matrix in homogeneous coordinates of an affine transformation has the same value as the determinant if it were a linear transformation! And for a linear transformation we know that the area of a transformed unit square is given by the absolute value of the determinant. So the same must be true for affine transformations. In a formula:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & t_1 \\ a_{21} & a_{22} & a_{23} & t_2 \\ a_{31} & a_{32} & a_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{vmatrix} =$$

$$0 \cdot \begin{vmatrix} a_{12} & a_{13} & t_1 \\ a_{22} & a_{23} & t_2 \\ a_{32} & a_{33} & t_3 \end{vmatrix} + 0 \cdot \begin{vmatrix} a_{11} & a_{13} & t_1 \\ a_{21} & a_{23} & t_2 \\ a_{31} & a_{33} & t_3 \end{vmatrix} + 0 \cdot \begin{vmatrix} a_{11} & a_{12} & t_1 \\ a_{21} & a_{22} & t_2 \\ a_{31} & a_{32} & t_3 \end{vmatrix} + 1 \cdot \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$